Pergamon

Int. J. Solids Structures Vol. 33, No. 29, pp. 4327–4336, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00232–4

FURTHER ANALYSIS OF END EFFECTS FOR PLANE DEFORMATIONS OF SANDWICH STRIPS

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(*Received* 27 *April* 1995; *in revised form* 8 *September* 1995)

Abstract—Saint-Venant end effects for plane deformations of sandwich strips are investigated. For two phase composite strips, with homogeneous isotropic layers that are subjected to self-equilibrated end loads, the exponential decay of end effects is characterized in terms of eigenvalues analogous to the Fadle-Papkovich eigenvalues for the homogeneous strip. An earlier study of this problem was carried out by Choi and Horgan (1978, Saint-Venant end effects for plane deformation of sandwich strips. *Int. J. Solids Structures* **14,** 187–195). Here we simplify and extend this analysis by using the results of Dundurs on reduced dependence on the elastic constants. The decay rate is shown to depend quadratically on the two Dundurs' constants α,β. Numerical results yield graphs for the decay rate in the α,β parallelogram, which are immediately applicable for design purposes. For any combination of materials in the two-phase sandwich strip, these plots allow one to determine the Saint-Venant decay length. For the case of a relatively compliant inner core, an asymptotic result shows that the characteristic decay length is much larger than that for a homogeneous isotropic strip. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The validity of the application of Saint-Venant's principle to plane elasticity problems for sandwich strips was first investigated analytically by Choi and Horgan (1978), motivated by earlier experimental studies of Alwar (1970). For two phase sandwich strips, with homogeneous isotropic layers and subjected to self-equilibrated end loads, the exponential decay of Saint-Venant end effects away from the loaded ends was characterized in terms of complex eigenvalues, analogous to the well-known Fadle-Papkovich eigenvalues for the homogeneous isotropic strip (see e.g. Horgan and Knowles (1983), Horgan (1989) for reviews on Saint-Venant's principle). Of particular interest in Choi and Horgan (1978) was the case of a relatively compliant inner core, where extremely *slow decay* of the end effects was established. In this case, an asymptotic formula for the decay rate was also obtained in Choi and Horgan (1978), (see eqn (4.1) of that reference), which provides a very accurate estimate for the exact decay rate (see Horgan (1982) for further discussion). Related finite element calculations have been reported by Rao and Valsarajan (1980), Dong and Goetschel (1982), Okumura *et al.* (1985) and Goetschel and Hu (1985).

The transcendental equations for the eigenvalues derived in Choi and Horgan (1978) were expressed in terms of *three* material parameters. In the Appendix to that paper, the authors suggest that the results may be simplified by exploiting the results of Dundurs (1967, 1969) regarding the reduced dependence on the elastic constants for plane deformations of two-phase isotropic composites. The purpose of the present paper is to investigate in detail the consequences of this simplification.

In general, when a composite body consisting of two homogeneous isotropic elastic phases is subjected to prescribed surface tractions, the stress field depends on *three* parameters involving the elastic constants, for instance the ratio of the shear moduli and the Poisson's ratios of the two materials. However it has been shown by Dundurs (1967, 1969) that when the composite body is in a state of plane deformation and when there are no net forces on internal boundaries, the stress field in the composite depends only on *two* combinations of the elastic constants. Although the choice of the two composite parameters is not unique, the constants α and β introduced by Dundurs have been universally accepted. The Dundurs' constants, which are measures for the mismatches in the uniaxial and bulk compliances, are defined by

$$\alpha = \frac{\Gamma(\kappa_1+1)-(\kappa_2+1)}{\Gamma(\kappa_1+1)+(\kappa_2+1)}, \quad \beta = \frac{\Gamma(\kappa_1-1)-(\kappa_2-1)}{\Gamma(\kappa_1+1)+(\kappa_2+1)}, \tag{1}$$

where $\Gamma = \mu_2/\mu_1$, $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$, $(i = 1,2)$ for generalized plane stress. Here μ_i, ν_i, $(i = 1,2)$ are the shear moduli and Poisson's ratios, respectively. Under the usual physical assumptions $\mu_i > 0$, and $0 \leqslant \nu_i \leqslant 1/2$, the admissible values of α and β are restricted to a parallelogram in the α,β plane, with the extreme values $-1 \leqslant \alpha \leqslant 1$, $-1/2 \leqslant \beta \leqslant 1/2$ (see for instance Fig. 2 in Dundurs, 1969). Identical materials are represented by $\alpha = \beta = 0$, and the case of equal shear moduli (i.e., $\Gamma = 1$) corresponds to the straight line $\alpha = \beta$. The extreme cases of $\Gamma = 0$ and $\Gamma \to \infty$ correspond to the straight lines $\alpha = -1$ and $\alpha = +1$ respectively. It is noted that a single point in the α,β plane corresponds to many material combinations. An interesting review of the correlation between α,β and the elastic moduli was given recently by Schmauder and Meyer (1992). In particular, the α,β values for a variety of metal/ceramic composites are given in this reference, showing that α,β normally fall within a narrow band in the parallelogram.

In the next section, we provide a brief summary of the problem formulated in Choi and Horgan (1978) and rewrite the transcendental equations for the eigenvalues using the two Dundurs' constants. The exponential decay rate for the stresses is given by the real part of the eigenvalue with the smallest real part. While the eigenvalue parameter appears in these transcendental equations in a complicated manner, these equations are *quadratic* in the parameters α and β. In Section 3, the numerical solution for the eigenvalues is discussed in detail, and the results plotted as loci of constant values in the α,β parallelogram. In particular, the decay rate for end effects is graphed in the α,β plane. The results are presented in a form immediately accessible to designers. Thus for *any combination of materials in the two-phase sandwich strip*, Figs 2(a) and 3(c)–7 allow one to determine the Saint-Venant decay length precisely. Any asymptotic result for the case of a relatively compliant inner core is also described.

Analogs of the results obtained here have also been obtained by Wijeyewickrema and Keer (1994) for a sandwich strip with slipping interfaces and by Wijeyewickrema (1995) for a multilayered composite.

2. FORMULATION OF THE PROBLEM AND CHARACTERIZATION OF THE DECAY RATE IN TERMS OF EIGENVALUES

The semi-infinite sandwich strip considered (see Fig. 1) consists of outer layers of identical homogeneous isotropic material with elastic constants μ_1, ν_1 and thickness $2c_1$ and the inner layer with elastic constants μ_2, ν_2 and thickness $2c_2$. The layers are perfectly bonded at the interfaces and the top and bottom surfaces are free of traction. The displacements and stresses in each layer can be obtained by means of the Fadle-Papkovich functions (Timoshenko and Goodier (1970), p. 62; Choi and Horgan (1978), eqn (2.10)) which exhibit exponential decay in the axial direction. They are given in the Appendix where the superscripts and subscripts 1, 2 refer to the outer layers and inner core respectively. By symmetry arguments, it is sufficient to consider the half strip as the basic geometric configuration.

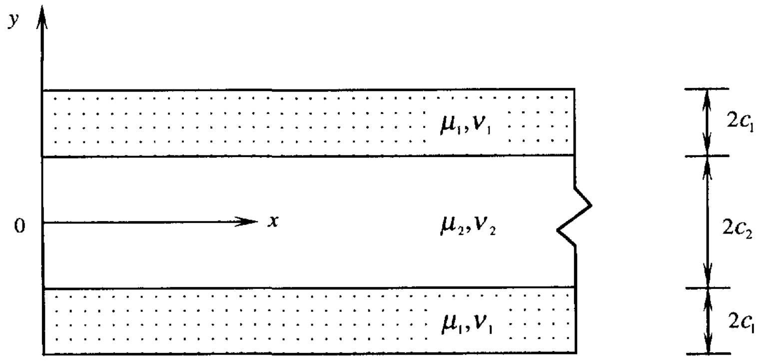

Fig. 1. Sandwich strip composite.

The interface conditions and the outer boundary conditions yield a system of six equations for the six unknown coefficients associated with the displacement and stress fields. The determinant of the coefficients is obtained for symmetric deformation of the core as

$$\Delta_S(\gamma;\alpha,\beta,f) = \tfrac{1}{2}\alpha^2[a_1-8(1-f)f^2\gamma^3] \\ +\alpha\{\beta[2f^2\gamma^2a_2-4\gamma(1-f)a_3]+[(1-2f^2\gamma^2)a_2-4\gamma(1-f)a_3]\} \\ +\{2\beta^2a_2a_3+\beta[2f^2\gamma^2a_2+4\gamma(1-f)a_3]+\tfrac{1}{2}a_4\}, \quad (2)$$

where

$$a_1 = \sin 2(1-2f)\gamma+2(1-2f)\gamma, \quad a_2 = \sin 2(1-f)\gamma+2(1-f)\gamma, \\ a_3 = \sin^2 f\gamma-f^2\gamma^2, \quad a_4 = \sin 2\gamma+2\gamma, \quad (3)$$

and for the anti-symmetric case as

$$\Delta_A(\gamma;\alpha,\beta,f) = \tfrac{1}{2}\alpha^2[b_1+8(1-f)f^2\gamma^3] \\ +\alpha\{\beta[2f^2\gamma^2b_2+4\gamma(1-f)b_3]+[(1-2f^2\gamma^2)b_2+4(1-f)b_3]\} \\ +\{2\beta^2b_2b_3+\beta[2f^2\gamma^2b_2-4\gamma(1-f)b_3]+\tfrac{1}{2}b_4\}, \quad (4)$$

where

$$b_1 = \sin 2(1-2f)\gamma-2(1-2f)\gamma, \quad b_2 = \sin 2(1-f)\gamma-2(1-f)\gamma, \\ b_3 = \sin^2 f\gamma-f^2\gamma^2 = a_3, \quad b_4 = \sin 2\gamma-2\gamma. \quad (5)$$

Here f is the volume fraction of the outer layers (given by eqn (3.1) of Choi and Horgan (1978)), that is, the ratio of the thickness of the outer layers to the total strip thickness:

$$f = \frac{4c_1}{4c_1+2c_2}. \quad (6)$$

The (generally complex) eigenvalue γ was introduced in eqn (3.3) of Choi and Horgan (1978) to compare the exponential decay rate for the layers with that for a homogeneous strip of the same total width

$$\frac{\gamma}{2c_1+c_2} = \frac{\gamma_1}{c_1} = \frac{\gamma_2}{c_2}. \tag{7}$$

We note that Δ_S and Δ_A have a complicated structure in the variable γ but are simply quadratic in the constants α,β. The equations $\Delta_S(\gamma;\alpha,\beta,f) = 0$ and $\Delta_A(\gamma;\alpha,\beta,f) = 0$ are identical to the transcendental equation for the eigenvalues γ obtained in eqn (3.5) of Choi and Horgan (1978). As in Choi and Horgan (1978), we confine attention in what follows to investigating the roots of the transcendental equations and thus, to analyzing the exponential decay rate for the stresses with axial distance from the end $x = 0$.

A complete solution to a boundary-value problem for the sandwich strip would require an eigenfunction expansion solution, with summation over the infinite set of eigenvalues (and eigenfunctions). The constants involved in such an expansion would depend on boundary conditions at $x = 0$. While the completeness of the analogous Fadle-Papkovich eigenfunctions for the single layer has been established by Gregory (1980), a corresponding result for the present problem is not known. We shall not pursue this question here, but rather observe that, *for arbitrary end loads*, the exponential decay rate for the stresses is the real part of the eigenvalue γ with the smallest real part (cf. the discussion in Horgan and Knowles (1983), pp. 231–232). If we isolate the decaying exponential term in the stresses as e^{-kx}, then the exponential decay rate k may be characterized as

$$k(\textit{strip thickness}) = 2\,\mathrm{Re}\,(\gamma) \tag{8}$$

where γ is the zero of eqn (2) or (4) with the smallest real part. The characteristic decay length is $1/k$.

3. DISCUSSION

The roots of $\Delta_S(\gamma;\alpha,\beta,f) = 0$ and $\Delta_A(\gamma;\alpha,\beta,f) = 0$ are determined numerically. For a given volume fraction f, when γ is real, a root is chosen and its locus is plotted in the α,β plane by solving the quadratic in α for each β in the range $-1/2 \leqslant \beta \leqslant 1/2$. When γ is complex, we express γ as $\gamma = \xi + i\eta$ and write

$$\Delta_S(\xi+i\eta;\alpha,\beta,f) = \Delta_S^R(\xi,\eta;\alpha,\beta,f) + i\Delta_S^I(\xi,\eta;\alpha,\beta,f) \tag{9}$$

and solve the two quadratics in α to obtain ξ, η curves. The $\eta = 0$ curves are obtained by satisfying

$$\Delta_S(\gamma;\alpha,\beta,f) = 0 \quad \text{and} \quad \frac{\partial \Delta_S}{\partial \gamma}(\gamma;\alpha,\beta,f) = 0 \tag{10}$$

simultaneously and are indicated in the figures by thick lines. A similar procedure is adopted to determine the roots of $\Delta_A(\gamma;\alpha,\beta,f) = 0$.

We now analyze the roots of $\Delta_S(\gamma;\alpha,\beta,f) = 0$ and $\Delta_A(\gamma;\alpha,\beta,f) = 0$. The roots of $\Delta_S(\gamma;\alpha,\beta,f) = 0$ are plotted in Figs 2–4 for $f = 0.2$, 0.5 and 0.8 in the α,β parallelogram,

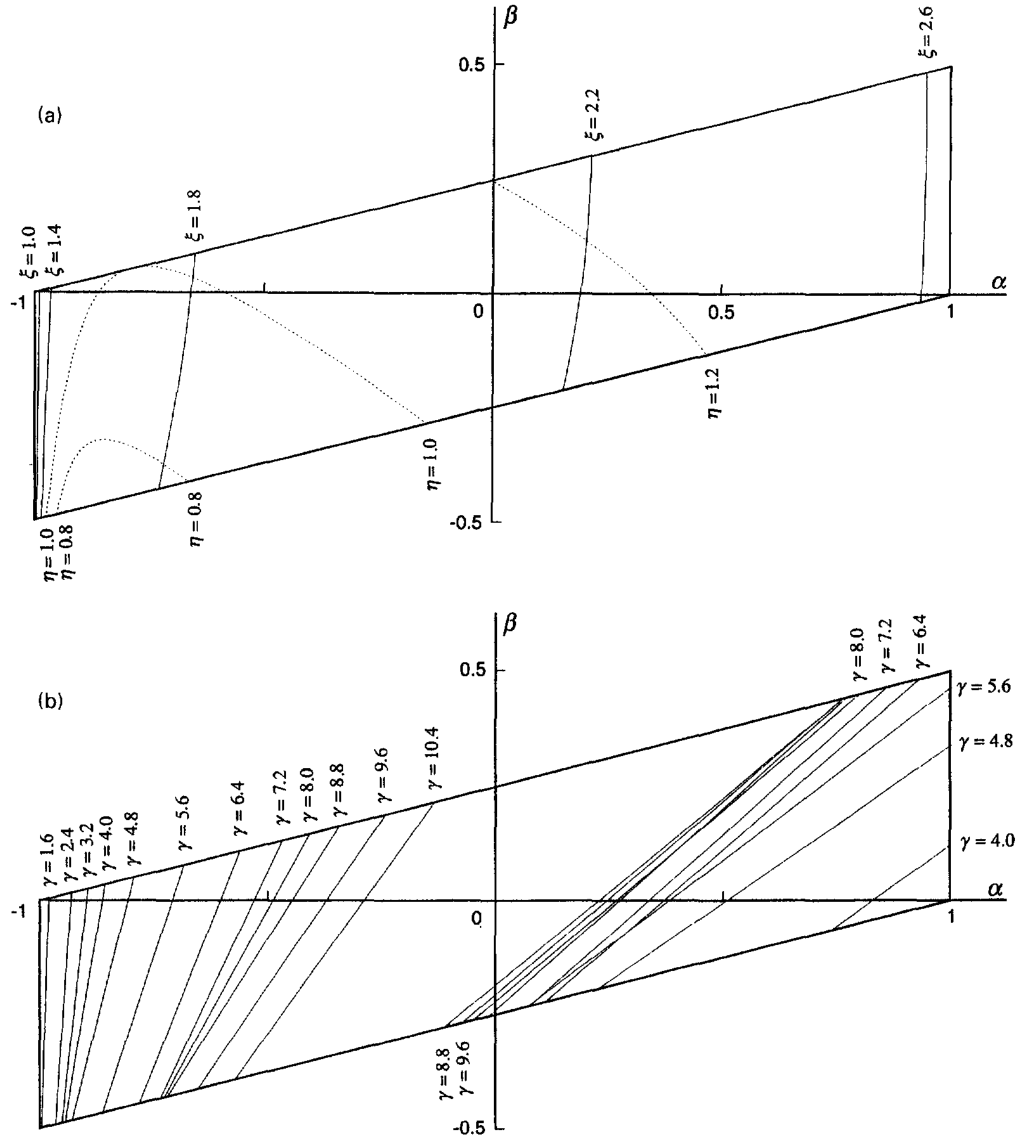

Fig. 2. (a) Roots of $\Delta_S(\gamma;\alpha,\beta,f) = 0$, for $f = 0.2$, complex roots $\gamma = \xi + i\eta$. (b) Roots of $\Delta_S(\gamma;\alpha,\beta,f) = 0$, for $f = 0.2$, real roots.

while roots of $\Delta_A(\gamma;\alpha,\beta,f) = 0$ are plotted in Figs 5–7 for the same values of the volume fraction. In these figures, the roots which correspond to the smallest positive real part of γ are shown. To arrive at these results, one needs to consider three subfigures, as illustrated in Figs 3(a)–(c) for the volume fraction $f = 0.5$, namely where: (a) the complex roots $\gamma = \xi + i\eta$ are plotted as contours ξ = constant, η = constant covering the whole α,β parallelogram, (b) real roots γ are plotted as contours γ = constant, and (c) roots which yield the smallest positive real part of γ are similarly plotted. Thus, in view of (8), the (c) figures *provide the exponential decay rates for the stresses*. There is no Fig. 2(c) since the complex roots always yield the roots of smallest real part when $f = 0.2$. Thus, in this case, the decay rates are given by the values of ξ in Fig. 2(a). A similar situation arises when $f = 0.1$ and 0.3 (see e.g. Fig. 2 of Choi and Horgan (1978)).

Consider a typical set of these figures, say Figs 3(a)–(c) in the symmetric case corresponding to a volume fraction $f = 0.5$. The intricate dependence of the eigenvalues on the material parameters α and β is striking. From Figs 3–4, we can see that the decay rates (for the symmetric case) correspond always to the *real roots* in the far right quarter of the

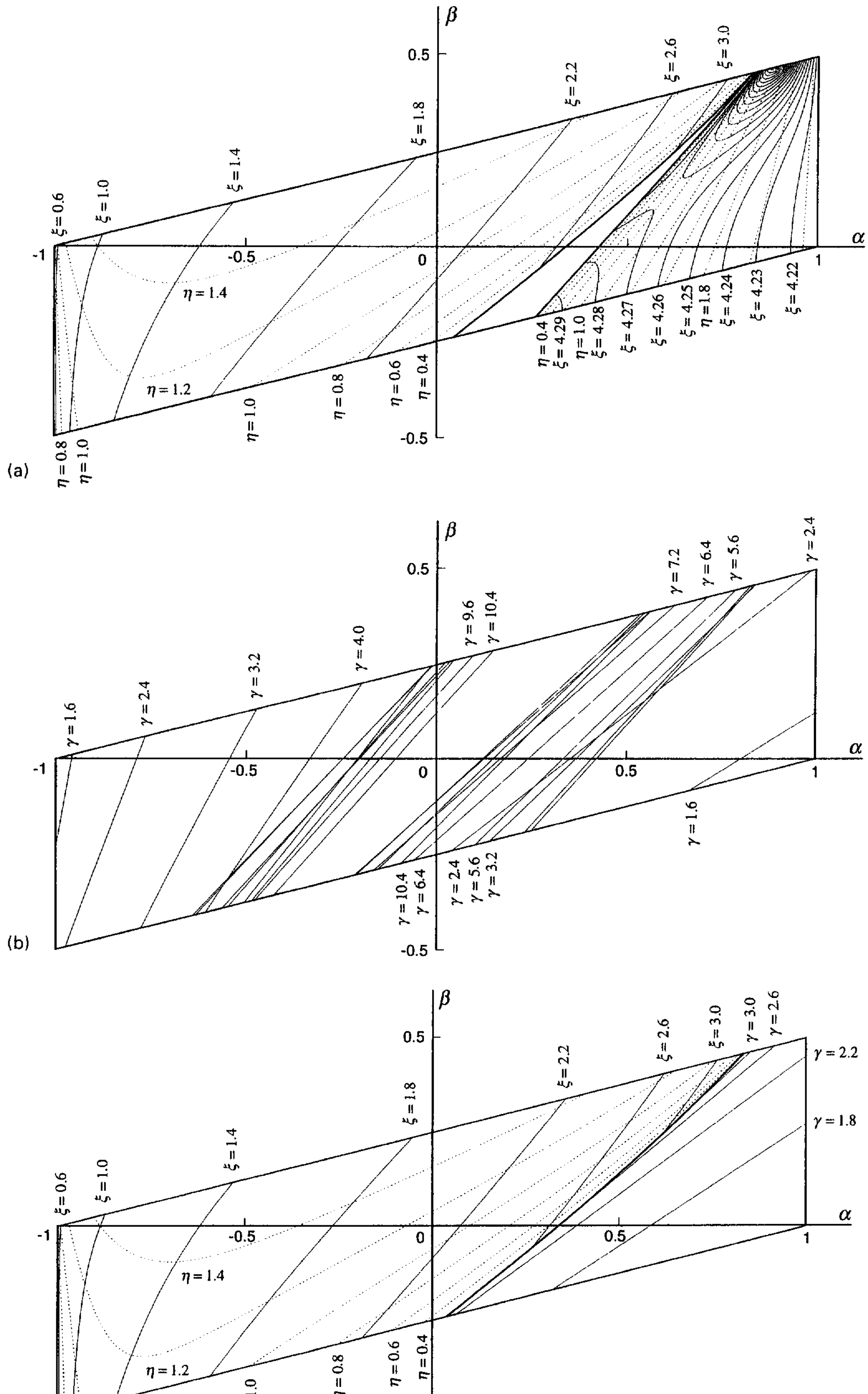

Fig. 3. (a) Roots of $\Delta_S(\gamma\,;\alpha,\beta,f) = 0$, for $f = 0.5$, complex roots $\gamma = \xi + i\eta$. (b) Roots of $\Delta_S(\gamma\,;\alpha,\beta,f) = 0$, for $f = 0.5$, real roots. (c) Roots of $\Delta_S(\gamma\,;\alpha,\beta,f) = 0$, for $f = 0.5$, which correspond to the smallest positive real part of γ.

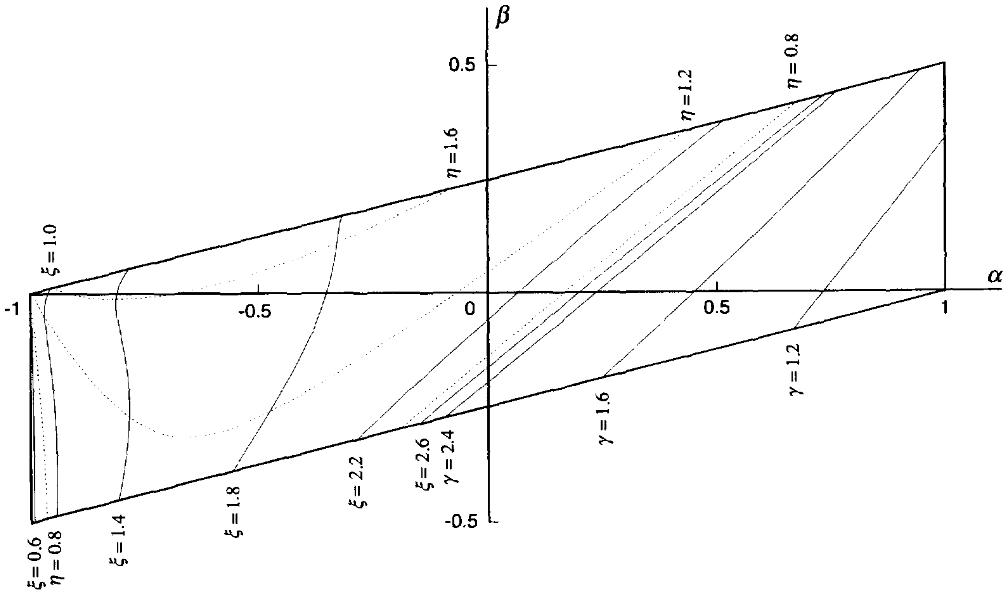

Fig. 4. Roots of $\Delta_S(\gamma; \alpha, \beta, f) = 0$, for $f = 0.8$, which correspond to the smallest positive real part of γ.

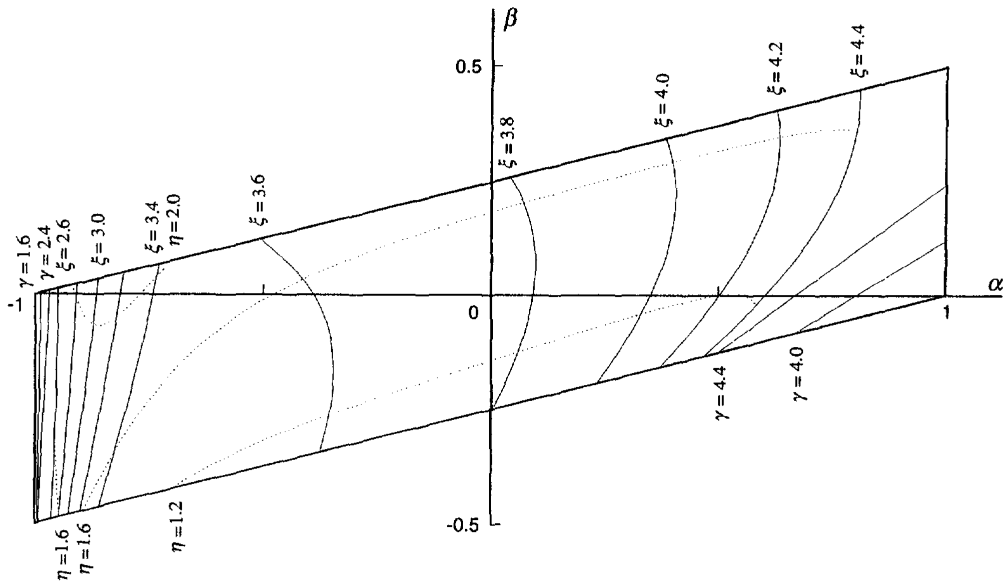

Fig. 5. Roots of $\Delta_A(\gamma; \alpha, \beta, f) = 0$, for $f = 0.2$, which correspond to the smallest positive real part of γ.

α,β parallelogram consistent with the results in Fig. 2 of Choi and Horgan (1978). The monotonicity of these rates with α (or β) may be examined from these figures. Consider, the example, Fig. 3(c) where $f = 0.5$. For *fixed* β, we see that the (symmetric) decay rate k given by (8) is monotone increasing in α until the relevant root is real, after which k is monotone decreasing in α. A similar behavior was exhibited in Choi and Horgan (1978), Fig. 2.

For arbitrary end loadings, it should be emphasized that the decay rate for the composite strip may be governed by the symmetric or the anti-symmetric case. By contrast, *for the homogeneous isotropic strip, the decay rate always comes from the symmetric eigencondition* (see e.g. Timoshenko and Goodier (1970), p. 62). Thus at the origin in the α,β

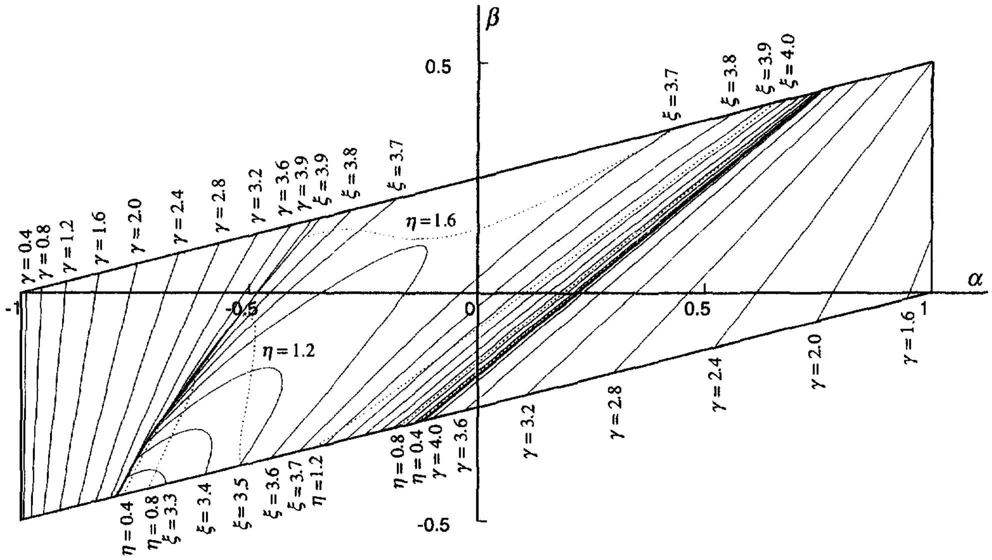

Fig. 6. Roots of $\Delta_A(\gamma\,;\alpha,\beta,f) = 0$, for $f = 0.5$, which correspond to the smallest positive real part of γ.

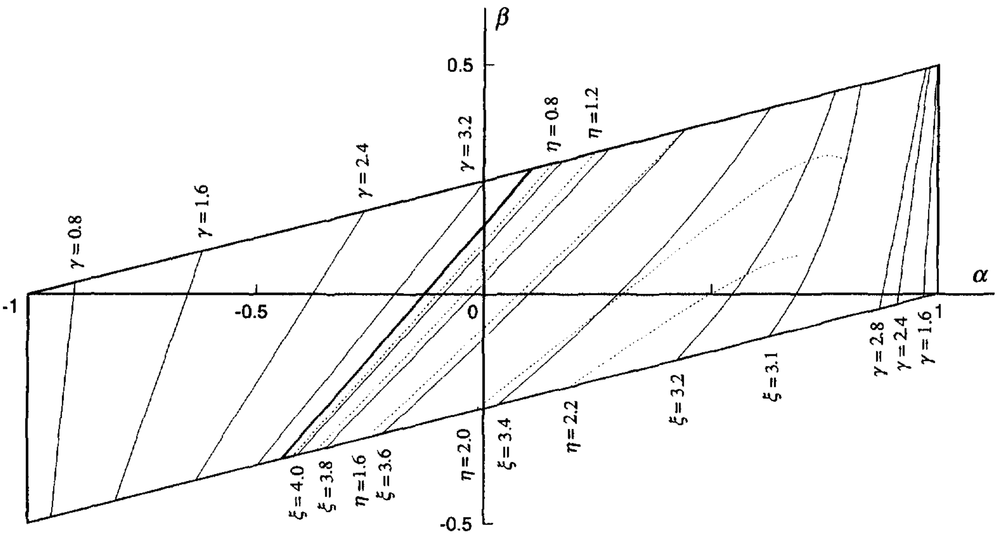

Fig. 7. Roots of $\Delta_A(\gamma\,;\alpha,\beta,f) = 0$, for $f = 0.8$, which correspond to the smallest positive real part of γ.

plane, (regardless of the value of f), we always have

$$\gamma = \xi + i\eta = 2.106 + i1.125, \tag{11}$$

in Figs 2–4. Now consider, for example, Fig. 3(c) and Fig. 6 where $f = 0.5$. On comparing these figures, it is seen that the decay rate k given by (8) comes from the symmetric case *except near* $\alpha = -1$ where the anti-symmetric case yields the smallest real γ (when $\beta = 0$). From Fig. 6, we find that, when $f = 0.5$, and $\beta = 0$, $\alpha \approx -0.95$, then $\gamma \approx 0.4$ so that

$$k \approx 0.8/(width) \tag{12}$$

which is about five times *smaller* than that for the homogeneous strip, on using (8), (11). Thus we obtain a *very slow decay* of end effects near $\alpha = -1$. We recall from Section 1 that this corresponds to $\Gamma \ll 1$, i.e.,

$$\mu_2 \ll \mu_1. \tag{13}$$

Thus, for a *relatively compliant inner core*, a slow decay of end effects is obtained. A similar result is exhibited in Fig. 3 of Choi and Horgan (1978).

This foregoing behavior holds, *regardless of the volume fraction f*. In fact was shown in Choi and Horgan (1978) (see eqn (4.1)) and Horgan (1982) (see eqn (5.3)) that

$$k \approx 2.3664\left(\frac{1+\alpha}{1-\alpha}\right)^{1/2}\left(\frac{f^2-3f+3}{f^3(1-f)}\right)^{1/2} \tag{14}$$

as $\alpha \to -1$ (for plane stress). The asymptotic result (14), which can be obtained from an asymptotic analysis of the smallest zero of eqn (4), thus predicts a very slow decay rate when $\mu_2 \ll \mu_1$. The result (14) agrees remarkably well with the exact decay rates computed in Choi and Horgan (1978) and in the present paper (see Horgan (1982) p. 420). For plane strain, a result of the form (14) also holds, where the right hand side of (14) has an additional multiplicative factor of $(1-\nu_2)^{1/2}$.

4. CONCLUDING REMARKS

The reduced dependence results of Dundurs for two phase composites have been shown to provide a particularly illuminating setting within which to analyze the decay of end effects in sandwich strips. For any combination of materials in the two phase sandwich strip, Fig. 2(a) and Figs 3(c)–7 allow one to determine the Saint-Venant decay length precisely.

Acknowledgements—The work of C.O.H. was supported by the U.S. Army Research Office under Grant DAAH04-94-G-0189, by the U. S. National Science Foundation under Grant MSS-91-02155 and by the U.S. Air Force Office of Scientific Research under Grant AFOSR-F49620-95-1-0308. The work of J.D. was supported by NSF under the Grant MSS-94-02285.

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APPENDIX

Using the abbreviations

$$x_1 = \gamma_1 x/c_1, \quad y_1 = \gamma_1 y/c_1 \tag{A1}$$

the stresses and displacements in the outer layers are given by

$$\sigma^1_{xx}(x,y) = e^{-x_1}(\gamma_1/c_1)[-A_1\cos y_1 - B_1(y_1\cos y_1 + 2\sin y_1) - C_1\sin y_1 + D_1(2\cos y_1 - y_1\sin y_1)], \tag{A2}$$

$$\sigma^1_{yy}(x,y) = e^{-x_1}(\gamma_1/c_1)[A_1\cos y_1 + B_1 y_1\cos y_1 + C_1\sin y_1 + D_1 y_1\sin y_1], \tag{A3}$$

$$\sigma^1_{xy}(x,y) = e^{-x_1}(\gamma_1/c_1)[-A_1\sin y_1 + B_1(\cos y_1 - y_1\sin y_1) + C_1\cos y_1 + D_1(y_1\cos y_1 + \sin y_1)], \tag{A4}$$

$$4\mu_1 u^1_x(x,y) = e^{-x_1}\{2A_1\cos y_1 + B_1[2y_1\cos y_1 + (\kappa_1+1)\sin y_1] + 2C_1\sin y_1 + D_1[-(\kappa_1+1)\cos y_1 + 2y_1\sin y_1]\}, \tag{A5}$$

$$4\mu_1 u^1_y(x,y) = e^{-x_1}\{2A_1\sin y_1 + B_1[(\kappa_1-1)\cos y_1 + 2y_1\sin y_1] - 2C_1\cos y_1 + D_1[-2y_1\cos y_1 + (\kappa_1-1)\sin y_1]\}, \tag{A6}$$

where A_1, B_1, C_1, D_1 are arbitrary constants.

With

$$x_2 = \gamma_2 x/c_2, \quad y_2 = \gamma_2 y/c_2 \tag{A7}$$

the stresses and displacements in the inner layer for symmetric deformation of the inner layer can be expressed as

$$\sigma^2_{xx}(x,y) = e^{-x_2}(\gamma_2/c_2)[-A_2\cos y_2 + D_2(2\cos y_2 - y_2\sin y_2)], \tag{A8}$$

$$\sigma^2_{yy}(x,y) = e^{-x_2}(\gamma_2/c_2)[A_2\cos y_2 + D_2 y_2\sin y_2], \tag{A9}$$

$$\sigma^2_{xy}(x,y) = e^{-x_2}(\gamma_2/c_2)[-A_2\sin y_2 + D_2(y_2\cos y_2 + \sin y_2)], \tag{A10}$$

$$4\mu_2 u^2_x(x,y) = e^{-x_2}\{2A_2\cos y_2 + D_2[-(\kappa_2+1)\cos y_2 + 2y_2\sin y_2]\}, \tag{A11}$$

$$4\mu_2 u^2_y(x,y) = e^{-x_2}\{2A_2\sin y_2 + D_2[-2y_2\cos y_2 + (\kappa_2-1)\sin y_2]\}, \tag{A12}$$

where A_2, B_2, C_2, D_2 are arbitrary constants, while the stresses and displacements in the inner layer for anti-symmetric deformation of the inner layer are given by

$$\sigma^2_{xx}(x,y) = e^{-x_2}(\gamma_2/c_2)[-B_2(y_2\cos y_2 + 2\sin y_2) - C_2\sin y_2], \tag{A13}$$

$$\sigma^2_{yy}(x,y) = e^{-x_2}(\gamma_2/c_2)[B_2 y_2\cos y_2 + C_2\sin y_2], \tag{A14}$$

$$\sigma^2_{xy}(x,y) = e^{-x_2}(\gamma_2/c_2)[B_2(\cos y_2 - y_2\sin y_2) + C_2\cos y_2], \tag{A15}$$

$$4\mu_2 u^2_x(x,y) = e^{-x_2}\{B_2[2y_2\cos y_2 + (\kappa_2+1)\sin y_2] + 2C_2\sin y_2\}, \tag{A16}$$

$$4\mu_2 u^2_y(x,y) = e^{-x_2}\{B_2[(\kappa_2-1)\cos y_2 + 2y_2\sin y_2] - 2C_2\cos y_2\}. \tag{A17}$$